Observations on non-commutative field theories in coordinate space

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Abstract

We discuss non-commutative field theories in coordinate space. To do so we introduce pseudo-localized operators that represent interesting position dependent (gauge invariant) observables. The formalism may be applied to arbitrary field theories, with or without supersymmetry.

The formalism has a number of intuitive advantages. First it makes clear the appearance of new degrees of freedom in the infrared. Second, it allows for a study of correlation functions of (composite) operators. Thus we calculate the two point function in position space of the insertion of certain composite operators. We demonstrate that, even at tree level, many of the by now familiar properties of non-commutative field theories are manifest and have simple interpretations. The form of correlation functions are such that certain singularities may be interpreted in terms of dimensional reduction along the non-commutative directions: this comes about because these are theories of fundamental dipoles.

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1 Introduction

A great deal of effort has recently gone into the study of field theories on non-commutative spaces, partly due to the realization that non-commutative geometries are an important aspect of D-brane physics [1, 2]. There are many interesting features, including a relation between UV and IR physics [3, 4, 5]. A basic problem in any field theory is that of identifying the physical observables. In the case of gauge theories on non-commutative spaces, this is of particular interest and there are proposals for a class of such operators built out of Wilson lines.[6, 7, 8, 9] As well, there have been many discussions of the renormalization properties of these theories (see for example Refs. [10, 4, 3, 11, 12]).

In this paper, we will have several goals. A primary motivation for us in thinking about non-commutative field theories has been to come to some understanding of the UV/IR correspondence. In particular, non-commutative field theories apparently force us to give up on our Wilsonian thinking that is so familiar in standard field theories. What should replace our Wilsonian intuition? If Wilsonian reasoning fails, in what sense can we demonstrate the renormalizability or even the existence of a field theory?

To begin such explorations, we build field theory observables directly in position space. In ordinary regulated field theories, local operators should be thought of as smeared over a region whose size is determined by the regulator. Beginning with this idea, we find that it is possible to carry it over to the non-commutative case. An advantage of this formalism is that it is straightforwardly extended to theories with gauge symmetry and supersymmetry. There are aspects to this problem which have been discussed previously; in particular, the non-commutative coordinates are covariantized with respect to the symmetry group[13, 14]. Thus in a non-commutative field theory, the notion of "local operator" is replaced by integrals over non-commutative space with covariantized smearing distributions. We will refer to these operators as *pseudo-local*.

Having constructed pseudo-local operators, one can consider the coordinate space correlation functions. These theories are characterized by the correlation functions of operators inserted at various positions, and since position is not an invariant concept in non-commutative field theories, we need to decide on a choice of correlation functions. It is natural to take these to correspond to the insertion of pseudo-local operators. We note that this is the most useful setup in order to study the gravity–field theory correspondence for these theories.[15, 16, 17]

As we stated, a motivation to understand these correlation functions comes from an attempt to define these theories at the quantum level. In a Wilsonian setup, a field theory is renormalizable if there is a UV fixed point of the renormalization group. Otherwise we just have an effective field theory which is valid below some scale, and all possible operators that are consistent with the symmetries of the theory are generated by integrating out the heavy modes. Given the non-commutativity scale as the origin for the UV/IR correspondence, one might wonder if this approach makes any sense at all beyond the non-commutativity scale. One may need new degrees of freedom in the ultraviolet. In the Wilsonian approach, these new degrees of freedom can be integrated out and their effects will be apparent in the effective action.

The existence of this UV fixed point may be studied by looking at the universality of the singularities in the correlation functions of composite operators. As these theories are non-commutative, one can expect that as field theories they should correspond to some new universality class in the UV. Our results show that these theories are truly non-local in the variables used to study them. We find singular behavior in correlation functions at large distances in the non-commutative directions and short distances in commutative directions, even in the free field limit. These are long-range correlations in the non-commutative directions. Given these correlations, one has a very hard time imagining how to perform a 'block spin' renormalization procedure. The geometric picture of the RG flow as exploring shorter distances on a lattice breaks down in the four dimensional sense. Thus, non-commutative field theories are not Wilsonian in a standard sense. The passage from coordinate to momentum space is non-trivial in the sense that large distance does not necessarily correspond to small momentum.

The paper is organized as follows. In Section 2 we describe how to construct gauge invariant observables which are position dependent. We then describe how to extend this formalism to the construction of chiral gauge invariant observables. Algebraic details may be found in an appendix. In Section 3 we describe how to construct multi-trace operators which are position dependent, and we comment on how to interpret effective actions with insertions of these operators. The net result is that they can not be considered to be 'local' actions, and to change them into this form it is necessary to introduce auxiliary fields. These auxiliary fields are determined by a commutative geometry associated to the non-commutative geometry, and thus may be interpreted as auxiliary closed string states.[2, 4]

Next, in Section 4, we discuss quantum aspects of the insertion of single trace operators, how certain divergences may be removed and how the single trace composite operators inevitably mix with multitrace operators (if we attempt to interpret renormalization in a Wilsonian setting).

In Section 5 we study the correlation functions of two simple operators inserted at different places. We show that the correlation functions can have ultraviolet divergences at long distances on non-commutative coordinates, and short distances in the commuting coordinates. These singularities are reminiscent of dimensionally reduced theories with a continuum spectrum of massive states (such as in a non-compact compactification).

2 Gauge invariant observables: an algebraic approach

In a commutative field theory, we are interested in gauge invariant observables; in particular, we usually think of inserting operators at specific values of the coordinates and thus build local operators. For non-commutative field theories, there is no invariant meaning of placing an operator at x, as the coordinates do not commute.

Let us begin with a few comments on commutative field theories. Even in a (regulated) commutative field theory, one cannot place operators closer than the scale at which the regularization is performed. It is more natural instead to take operators which do not have δ-function support at the point, but are rather smeared over a non-locality scale set by the

regulator, and they are thus insensitive to the ultraviolet beyond the regularization scale. A natural operator defined near a point y in a regulated theory, $\tilde{\mathcal{O}}(y)$, is thus understood as a convolution of an operator $\mathcal{O}(x)$ with a distribution f whose support is of a scale set by the regulator. Hence we write

$$\tilde{\mathcal{O}}(y) = \int dx \ \mathcal{O}(x) f(x - y) \tag{1}$$

with f a normalized (smooth) distribution. Removing the regulator is equivalent to localizing f towards a distribution with δ-function support.

Now, this construction may be carried over directly to non-commutative field theories where x is non-commutative: the allowable functions of x are the smooth distributions, and there is an integration operation. We introduce a distribution f of compact support (vanishing sufficiently fast at infinity), defined through a power-series expansion and an ordering prescription. The resulting operators will be referred to as pseudo-localized, or simply by the acronym *plops*.

For gauge field theories, the above procedure in non-commutative variables does not work directly as x is not gauge invariant, but may be modified very simply. We simply replace x_j by $\hat{x}^j = x^j + b^{ij} A_j(x)$, which transforms covariantly under gauge transformations (see the appendix for a more complete discussion[1]). Notice that the x are unbounded operators and A is compact, so the spectrum of $\hat{x}$ is essentially the spectrum of x perturbed by a small amount. Thus, any integral which converges for x will have the same asymptotics as the integral written with $\hat{x}$ replacing x.

Now since $\hat{x}$ transforms covariantly under gauge transformations, we can replace

$$f(x - y) \rightarrow f(\hat{x} - y) \tag{2}$$

Here y is a parameter and is taken as a commutative variable. It is a parameter in the distribution formal power series, and may be moved around by translation $x \rightarrow x - a$. Since $\hat{x}$ transforms in the adjoint of the gauge group, the above procedure is well defined if we also trace over the gauge group after inserting $f(\hat{x} - y)$. All operators which we can integrate are then in the adjoint of the gauge group. The definition of a pseudo-local operator is thus

$$\tilde{\mathcal{O}}_f(y) = \int dx \ \mathrm{tr} \ \mathcal{O}(x) f(\hat{x} - y) \tag{3}$$

for any $\mathcal{O}(x)$ which transforms in the adjoint representation, and any choice of distribution f. (The $\star$-product should be understood throughout.) We will also occasionally refer to these operators as single-trace.

Now, in order to define these operators, we have had to make a choice of distribution f. The construction has the virtue that in the limit $b \rightarrow 0$, we recover the gauge invariant commutative plops, and changing f will amount to a change of regularization in the resulting commutative field theory. In general, there is an issue of overcompleteness. In practice, one

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[1]This replacement has been also discussed elsewhere; see Refs. [13, 14].

would make some suitable choice of distributions f, perhaps chosen to be as localized as possible (such as a coherent state or δ-function distribution).

We have discussed here the construction of gauge-invariant operators in position space. It is important to compare this construction to the other examples of gauge invariant observables that are known, such as those involving open Wilson loops [6, 7, 8, 9]. In fact, the Wilson line construction is a special case of the above formalism, where we take[2]

$$f_k(y) = \exp(ik \cdot y). \tag{4}$$

Notice that f_k is maximally non-local and gives rise to the Fourier transform necessary for a formulation in momentum space. For many applications however, it is more appropriate to work directly in coordinate space. We will see several such applications later in this paper. In the following subsection, we note that the formalism may be carried over in a simple fashion to chiral superspace.

2.1 Supersymmetry and Chiral observables

In this section, we demonstrate that the formalism presented above may be supersymmetrized. A chiral state is annihilated by half of the supersymmetries, say $\bar{Q}$. In ordinary superspace, recall that the function $x^{\mu-} = x^\mu + i\theta\sigma^\mu\bar{\theta}$ is chiral, as the variation of x^μ under supersymmetry is compensated by the second term. Thus any power series in x^-, θ is annihilated by $\bar{D}$. This may be extended to pseudo-local operators by replacing x by x^-; if the operator $\mathcal{O}$ is chiral, then the integral

$$\tilde{\mathcal{O}}(y, \theta, \bar{\theta}) = \int dx\ \mathcal{O}(x, \theta, \bar{\theta}) f(y - x^-) \tag{5}$$

will be chiral. This is because $\bar{D}$ acts as a derivation, and thus satisfies the Leibnitz rule.

Now, we would like to produce chiral gauge invariant observables in non-commutative field theories. For the non-commutative case we had to modify x to make a gauge covariant operator $\hat{x}$. A naive replacement of $x^- \rightarrow x^- + b \cdot A$ gives us a gauge covariant operator, but we lose the chirality of the integral. However, we may build a superspace function $\hat{x}^-$ which is both gauge covariant and chiral by a simple additional modification. We will present the result in the constrained superfield formalism.[18]

We will study this by first introducing conjugate variables so that $\bar{D}$ may be written in terms of a commutator. This is quite natural in the case of non-commutative geometry. The conjugate spinors will be referred to as $\eta_\alpha, \bar{\eta}_{\dot{\alpha}}$, and satisfy

$$\{\eta_\alpha, \theta^\beta\} = i\delta_\alpha^\beta \tag{6}$$

$$\{\bar{\eta}_{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\} = i\delta_{\dot{\alpha}}^{\dot{\beta}} \tag{7}$$

They (anti)-commute with all the other (extended) superspace variables.

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[2]We thank A. Hashimoto for raising this question and helping us in the computation. See also Ref. [14].

The superspace chiral derivatives become

$$D_{\dot{\alpha}}\Phi = -[(\bar{\eta}_{\dot{\alpha}} + i(\theta\sigma^\mu)_{\dot{\alpha}}p_\mu), \Phi]_\pm, \quad D_\alpha = -[(\eta_\alpha + i(\sigma^\mu\bar{\theta})_\alpha p_\mu), \Phi]_\pm \tag{8}$$

Let us write the spinors on the right hand side of eqs. (8) as $\bar{\eta}'_{\dot{\alpha}}$, η'_α. Chiral operators are defined as those which commute with $\bar{\eta}'_{\dot{\alpha}}$. The $\bar{\eta}'$ do not commute with x, but they do commute with $x^{\mu-} = x^\mu + i\theta\sigma^\mu\bar{\theta}$ and with θ.

Covariantization procedes by introducing gauge superfields

$$\hat{\eta}'_\alpha = \eta'_\alpha + \Gamma_\alpha, \qquad \hat{\bar{\eta}}'_{\dot{\alpha}} = \bar{\eta}'_{\dot{\alpha}} + \bar{\Gamma}_{\dot{\alpha}} \qquad \hat{p}_\mu = p_\mu + \Gamma_\mu \tag{9}$$

These are constrained by requiring that the algebra is unmodified

$$\{\hat{\bar{\eta}}'_{\dot{\alpha}}, \hat{\bar{\eta}}'_{\dot{\beta}}\} = \{\hat{\eta}'_\alpha, \hat{\eta}'_\beta\} = 0, \qquad \{\hat{\eta}_\alpha, \hat{\bar{\eta}}'_{\dot{\alpha}}\} = -2\sigma^\mu_{\alpha\dot{\alpha}}\hat{p}_\mu \tag{10}$$

We will solve these constraints directly. In the non-commutative case, we must also covariantize

$$\hat{x}^{\mu-} = x^{\mu-} + b^{\mu\nu}\tilde{\Gamma}_\nu \tag{11}$$

We will require that this is chiral

$$[\hat{\bar{\eta}}'_{\dot{\alpha}}, \hat{x}^{\mu-}] = 0 \tag{12}$$

and gauge covariant.

We will work at linearized order, although it is straightforward to generalize. The constraints (10) may be solved by

$$\Gamma_\alpha = [\eta'_\alpha, \Omega] \qquad \Gamma_{\dot{\alpha}} = [\bar{\eta}'_{\dot{\alpha}}, \bar{\Omega}] \tag{13}$$

We then find

$$\Gamma_\mu = -i\partial_\mu\bar{\Omega} + \frac{1}{4}\bar{\sigma}^{\dot{\alpha}\alpha}_\mu[\bar{\eta}'_{\dot{\alpha}}, [\eta'_\alpha, \Omega - \bar{\Omega}]] \tag{14}$$

These are standard results and are unmodified here.

The chirality constraint (12) has solution

$$\tilde{\Gamma}_\nu = -i\partial_\nu\bar{\Omega} + \Phi_\nu \tag{15}$$

where Φ_μ is covariantly chiral. The superfield Φ_μ may be chosen such that the lowest component of $\tilde{\Gamma}_\mu$ is A_μ.

These results may be presented in the Wess-Zumino gauge; another possibility is to use a complexified gauge transformation to set $\bar{\Omega} = 0$, the chiral gauge. In this gauge, the chirality constraint is unmodified, and we find that $\hat{x}^{\mu-} = x^{\mu-} + b^{\mu\nu}a_\nu(x^-, \theta)$, explicitly chiral. This is covariant under a restricted class of chiral gauge transformations. In more general gauges, a gauge transformation is required to bring $\hat{x}^-$ back to this form.

These results should allow calculations to be done in supersymmetric non-commutative field theories. The single trace operators presented above are expected to correspond to supergravity fields in the large N limit, by the AdS/CFT [16]. Also, the formalism is important in that the renormalization properties, and existence, of interacting supersymmetric theories are expected to be in better shape.

3 Multi-Trace Operators and "Effective" Actions

The operators that we have discussed above are by construction single trace. We can algebraically construct multi-trace pseudo-local operators by taking products of the single trace plops. These are given by

$$\tilde{\mathcal{O}}_{MT,f_1,f_2,\dots f_n}(y) = \tilde{\mathcal{O}}_{1f_1}(y)\tilde{\mathcal{O}}_{2f_2}(y)\dots\tilde{\mathcal{O}}_{nf_n}(y) \tag{16}$$

If written in terms of integrals over the non-commutative space, this operator looks non-local as we get multiple integrals, one for each trace.[3] This discussion is at this point classical, but we will return to quantum issues later.

It is interesting to contemplate the fate of a Wilsonian formalism in this coordinate space language. To this end, we might suppose that we have a notion of an effective action at a given momentum scale. As we evolve this scale, the generic form of the action will involve an infinite set of plops, *including multi-trace operators*. This is inevitable: we expect this in ordinary gauge theories and therefore in non-commutative gauge theories as well. In fact, these multi-trace operators are the source of the UV-IR correspondence and the apparent loss of a Wilsonian action.

The point is as follows. Since the effective action is non-local (in the strongest sense we are defining locality by requiring the number of non-commutative integrals to be equal to one), it is natural to attempt to make it look local at the expense of introducing new degrees of freedom. In the case of double-trace operators, this is particularly simple: the degrees of freedom are "wormhole parameters." A simple pseudo-local double-trace operator takes the form

$$\int dy\tilde{\mathcal{O}}_{12}(y) = \int dy \int dx\mathcal{O}_1 f_1(\hat{x} - y) \int dx\mathcal{O}_2 f_2(\hat{x} - y) \tag{17}$$

We may introduce auxiliary fields as Lagrange multipliers in the path integral such that the action can be written

$$\int dyb_1(y)b_2(y) - i\int dyb_1(y)\tilde{\mathcal{O}}_1(y) - i\int b_2(y)\tilde{\mathcal{O}}_2(y) \tag{18}$$

and a functional integral over the $b(y)$ is understood. Notice that the b are functions of commutative variables y, and one can think of $\int dyb_1(y)\hat{f}(x-y) = \tilde{f}_b(\hat{x})$ as a new distribution with which we are convoluting the operator $\mathcal{O}_1$. As the b see a commutative geometry and as they couple to single trace operators in the field theory, one can think of them as closed string auxiliary fields by combining the ideas of [16, 2]. This result matches nicely with the infrared divergences encountered in [4, 3].

Thus the action for double-trace operators is of the form

$$\int dyb_1(y)b_2(y) + \int dx\mathcal{O}_1(x)\tilde{f}_{1,b_1}(\hat{x}) + \int dx\mathcal{O}_1(x)\tilde{f}_{2,b_2}(\hat{x}) \tag{19}$$

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[3] One can construct more general multitrace pseudolocal operators by further smearing the relative insertion of the operators, but the spirit of the construction of operators is the same.

which looks local from the non-commutative field theory perspective as having one integral over non-commutative space, and hence can be interpreted as a 'local' action. Notice that in this manner, operators which we would have called non-local can be made to look local in the sense of having only one integral over the non-commutative plane at the expense of introducing new degrees of freedom.

For triple trace operators and beyond one can add extra polynomial terms in the $b(y)$. Upon integrating out the auxiliary fields we will be left with a power series in multitrace operators.

4 Understanding composite plops

The analysis we have done so far is classical: the operators $\tilde{\mathcal{O}}(y)$ need to be defined at the quantum level. In the definition of $\tilde{\mathcal{O}}(y)$ we made the substitution $x \rightarrow \hat{x}$, which involves the connection. As the operators are all inserted at the same point (since in a single trace operator we only integrate over one variable) we expect to find divergences that would normally be associated to contact terms. The structure of these divergences is also apparent in the work of [9] where they were understood in terms of the diagrammatic expansion of the Wilson loop in a power series in the connection. Thus to define these operators quantum-mechanically we need to regularize the operators in some scheme. (In supersymmetric field theories these difficulties might be avoided if we use operators which are protected by supersymmetry. The formalism of Section 2.1 would be helpful in such computations.)

There are really two separate issues here. We can distinguish the singularities of the operator $\mathcal{O}$ (perhaps fixed via a normal ordering procedure) from that of the singularities between two single-trace operators.

In this section we will try to understand the structure of plops which have self-contractions. This is necessary if we want to understand the renormalizability of a bare action and the structure of counterterms required to render a theory finite at a given order in perturbation theory. There is a point to be mentioned here which is important. The connection enters in the definition of the operators, and if we want to calculate Feynman diagrams when we contract the connection, the propagators carry a factor of g^2. Thus, if we are renormalizing by taking power series in the couplings, only a finite number of contractions are needed at each order in g. For our calculation we will take the leading tree level contributions, and hence the connection will not play any role at all.

We begin by considering the operator $\phi^2(y)$. In a commutative theory, we might define this by point-splitting, subtracting the short-distance pole. More generally, we can write

$$\phi(y)^2 = \int dz \ K(z, y)\phi(z)\phi(y) \tag{20}$$

for some suitable kernel K. For example, we could choose the kernel such that the short-distance singularity is integrable. In the non-commutative case, we interpret this as a definition of $(\text{tr}\phi)^2$.

As for $\mathrm{tr}\phi^2$, we would, at least classically, have written

$$\phi^2(y) \;=\; \int dx\; \phi^2(x) f(x-y) \tag{21}$$

$$=\; \int dx\; \int dk_1\; dk_2\; e^{ik_1\cdot x} e^{ik_2\cdot x} \phi(k_1)\phi(k_2) f(x-y) \tag{22}$$

We propose that quantum mechanically, we write the same expressions. With some manipulations, we have

$$\phi^2(y) = \int dx\; \int dk_1\; dk_2\; e^{i(k_1+k_2)\cdot x} e^{ik_1 \wedge k_2} f(x-y)\phi(k_1)\phi(k_2) \tag{23}$$

As momenta are commutative parameters, we can subtract the pole in $\phi(k_1)\phi(k_2)$ as in commutative field theory. Thus we write

$$\phi^2(y) := \int dx\; \int dk_1\; dk_2\; e^{i(k_1+k_2)\cdot x} e^{ik_1 \wedge k_2} f(x-y) : \phi(k_1)\phi(k_2): \tag{24}$$

It is important to note that composite operators thus defined already contain Moyal phases.

Other composite operators may be normal-ordered in a similar fashion and include in general both planar and non-planar subtractions. For example, consider the composite pseudo-local operator $\phi^4(y)$ (within a ϕ^4 theory). We start with

$$\phi^4(y) \;=\; \int dk_1\; dk_2\; dk_3\; dk_4\; \mathrm{tr}\left[\phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4)\right] \tag{25}$$

$$\cdot\;\; e^{i(k_1+k_2+k_3+k_4)\cdot y}\; e^{ik_1\wedge k_2 + i(k_1+k_2)\wedge k_3 + i(k_1+k_2+k_3)\wedge k_4} \tag{26}$$

For simplicity we will take the distribution f to be a δ-function.

After a little work, we find

$$\mathrm{tr}(\phi^4)(y) \;=\; : \mathrm{tr}(\phi^4(y)): + \text{planar contractions}$$
$$+ \int d^d k \left(\frac{1}{k\circ k + 1/\Lambda^2}\right)^{1-d/2} \mathrm{tr}(\phi(k))\mathrm{tr}(\phi(y+b\cdot k))e^{ik\cdot y} \tag{27}$$

The planar terms are proportional to $: \mathrm{tr}\phi^2(y):$. In the non-planar term that we have written explicitly, we have dropped finite numerical constants for simplicity and taken the mass $m\to 0$.

The point that we wish to emphasize here is that double-trace operators necessarily appear in the definition of composite operators. If we attempted to interpret these equations as the starting point for a renormalization group approach to composite operators, then we conclude that single trace plops mix with multi-trace plops. This same behaviour occurs if we attempt to construct a Wilsonian action–we are unable to write a local, single-trace action.

5 Correlation functions of plops

In this section, we will consider tree level correlation functions of pseudo-local operators. We will find that much of what is known about non-commutative field theories is already visible at tree level–one does not need to look at loop effects to uncover the relevant physics. We will consider correlation functions $\langle \phi^n(0)\bar{\phi}^n(y)\rangle$.[4]

Let us first consider the correlation function with $n = 1$, $\langle \tilde{\phi}(0)\tilde{\bar{\phi}}(y)\rangle$. A simple computation shows that this is planar, and thus is the same in both commutative and non-commutative theories.

Now consider the composite operator $\phi^2(0)$. As discussed in the last section, we have

$$\phi^2(0) = \int dk_1 \ dk_2 \ e^{ik_1 \wedge k_2} \phi(k_1)\phi(k_2) \tag{28}$$

Now we want to calculate the correlation between this operator and $\bar{\phi}^2(y)$ which is similarly defined:

$$\bar{\phi}^2(y) = \int dk_1' \ dk_2' \ e^{ik_1' \wedge k_2'} \ \bar{\phi}(k_1')\bar{\phi}(k_2') \ e^{i(k_1' + k_2')\cdot y} \tag{29}$$

Clearly, the correlation function has two distinct contractions. One of these is planar while for the non-planar contraction we find:

$$\int dk_1 \ dk_2 \ \frac{e^{2ik_1 \wedge k_2}e^{-i(k_1+k_2)\cdot y}}{(k_1^2 + m^2)(k_2^2 + m^2)} \tag{30}$$

Regularizing,[5] we can rewrite this as

$$\int d\alpha \ d\beta \int dk_1 \ dk_2 \ e^{2ik_1 \wedge k_2}e^{-i(k_1+k_2)\cdot y}e^{-\alpha(k_1^2+m^2)}e^{-\beta(k_2^2+m^2)}e^{-\frac{1}{\Lambda^2 \alpha}-\frac{1}{\Lambda^2 \beta}} \tag{31}$$

Now, we do the integrals over k_1, k_2, and we separate the distance into $a = (a_{NC}, r)$, to distinguish separation along the non-commutative directions and along the commutative directions. We will assume that there are precisely two non-commutative directions; the generalization is straightforward. As the integral is Gaussian, it is easy to perform and (modulo numerical factors) gives

$$\int d\alpha d\beta \frac{1}{\alpha\beta + b^2} \frac{1}{(\alpha\beta)^{(d-2)/2}} e^{-\alpha m^2 - \beta m^2 - \frac{\alpha+\beta}{\alpha\beta+b^2}\frac{a_{NC}^2}{4} - \left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\left(\frac{r^2}{4}+\frac{1}{\Lambda^2}\right)} \tag{32}$$

As $b \to 0$ while holding Λ fixed, this goes smoothly to the commutative field theory limit. For finite b, we can examine the behavior of the integral using a saddle-point approximation,

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[4]Again, we will simplify the computations by taking $f(x - y) = \delta(x - y)$. Also, we now consider complex scalar fields.

[5]As is standard, the short distance regularization is done by modifying propagators $\frac{1}{k^2+m^2} \rightarrow \int_0^\infty d\alpha e^{-\alpha(k^2+m^2)-1/(\Lambda^2\alpha)}$.

and consider various limits. From the form of the integral it is clear that Λ and $1/r$ behave the same way: thus large momenta and small commutative distance are the same. Distance in the non-commutative directions behaves quite differently however. In particular, at large b, a_{NC} no longer acts like a UV cutoff, but instead $a_{NC}^2/4b^2$ behaves as a mass.

The nature of the saddle point depends on whether α, β are large or small with respect to the non-commutative parameter. If α, β are large, then b is not important for the evaluation of the saddle point, and the result reproduces the commutative field theory limit. This is true if $\frac{r^2}{4} + \frac{1}{\Lambda^2}$ is large in units of the non-commutative distance.

If α, β are very small, then b is important, and the relevant saddle point is obtained by the Taylor expansion of $\frac{\alpha+\beta}{\alpha\beta+b^2}\frac{a_{NC}^2}{4}$ around $\alpha, \beta \sim 0$, and we get instead a contribution to the mass which is proportional to a_{NC}^2. The asymptotic form of the expression is proportional to

$$\langle\phi^2(0)\bar{\phi}^2(a,r)\rangle \sim \frac{e^{-2r\sqrt{m^2+a^2/4b^2}}}{b^2 r^{d-4}} \tag{33}$$

Thus for r small (even when a_{NC} is very large) the correlation function has singular behaviour no matter what the value of a_{NC} is, although it is suppressed as we let a_{NC} get larger.

This singularity is like the one corresponding to a $d-2$ dimensional field theory, as if the non-commutative directions are compactified, and with an effective mass given by $m_{eff}^2 = m^2 + a^2/4b^2$. The fact that this singularity is present signals that there are long-distance correlations along the non-commutative directions. In an ordinary field theory long-distance correlations are attributed to massless modes propagating in these directions, thus it is not surprising that when we do loop diagrams we get 'infrared' singularities that would normally be associated to the effects of propagation of such modes.

The effective two-dimensional mass associated to a mode with momentum k along the non-commutative directions is $m^2 + k^2$. Thus the behavior of the correlation function is associated to a mode such that $bk \sim a$. Indeed, for the rigid dipole picture [10, 19] of the non-commutative modes, the effective distance between the ends of the dipoles at such momentum between points zero and a can be approximately zero. This is how we can interpret the above singularity. The UV degrees of freedom are large, they grow in size with the momentum. Thus localized (regularized) operators become larger in size as we send the cutoff scale to infinity.

It should be noted that the theory is more complicated than a simple dimensional reduction however. There are still the planar graphs which behave like ordinary particle graphs in d dimensions. The correct statement is that the degrees of freedom $\phi(k)$ are dipoles.

This is perhaps the clearest way in which we can see that the theory fails to be Wilsonian. As we increase the momentum, we are not necessarily exploring shorter distances, so the connection between the regulator and the coarse graining of the geometry fails. The structure of singularities fails to be universal in the standard sense. We find UV divergences at long distances, where we expect only IR physics.

Notice that the above calculation is independent of whether the theory is supersymmetric or not. Even with a lot of supersymmetry, these strong correlations will not disappear;

instead the behavior of the correlation functions will be protected by supersymmetry. If we take the cutoff to be related to a very small momentum scale with respect to the non-commutativity scale, the effects of the non-commutative parameter are unimportant, and the theory behaves like ordinary d-dimensional field theory. That is, the $\alpha, \beta \to 0$ singularity is cut off by the regulator scale, and as α, β are large compared to b, b is not important. In this sense we can recover ordinary field theory results in the infrared if we take the momentum cutoff scale to zero. This is in accordance with the supergravity solutions corresponding to $N = 4, d = 4$ SYM on a non-commutative plane [20, 21].

Clearly, the interesting behavior is for large Λ, when one is trying to understand the ultraviolet structure of the theory. The main result here is that the singularity structure makes part of the theory behave like a dimensionally reduced field theory.

If one is to do a Wilsonian analysis at all, it might be best to consider it only along the directions that are Lorentz invariant. The non-commutative directions should be treated as being compactified on a non-compact space, in a spirit similar to the Randall-Sundrum scenarios [22], as the mass parameter a is a continuous one. In this vein notice that for four dimensions, we get a singularity that behaves like $\log(r)$ at short distances. If we integrate over r to make a term in the effective action the measure will render these terms finite, so one can still call the theory renormalizable, but not in a Wilsonian interpretation.

6   Conclusion

In this paper we have given a procedure to build gauge invariant operators on a non-commutative field theory where the operators are position dependent, that is, they have a local character. We have also extended this construction to build chiral pseudo-local gauge invariant observables in supersymmetric non-commutative field theories.

The construction of states gives an overcomplete basis of operators, and also allows us to build multi-trace pseudo-local observables. In other treatments of these multi-trace operators higher $\star_n$ products are introduced[23, 24, 25, 26]. We feel that the presentation given here is intuitively more appealing, although it is not clear that one can write an effective action economically. This depends on the choice of basis for the functions which one uses in the convolutions.

We have studied some correlation functions of these operators at the free field theory level. Already these exhibit the UV/IR correspondence. We have found that there are 'non-planar' contributions to correlation functions of operators which are separated by large distances in the non-commutative directions and short distance in the commutative directions which give rise to UV singularities. These behave as if the theory is (partially) dimensionally reduced. As such, the theories fail to behave like standard Wilsonian field theories, the structure of UV singularities in correlation functions of operators do not conform to the geometrical picture of block-spin renormalization. The singularities in principle can give rise to infinities and renormalization of multi-trace operators when we integrate over the position of these operators, but we have found no such behavior in four dimensions. For higher dimensional field theories one can get UV singularities this way, but the theories are

already non-renormalizable and one needs to introduce new physics in the ultraviolet of the theory in any case.

Given the complicated structure of the correlation functions, one might still ask if there is some other universal singular behavior in the correlation functions which might give us some operational definition of non-commutative field theories and their renormalizability. This new structure might account for a new way to understand operator product expansions and rescue in some sense the Wilsonian approach to renormalization, which has proved so useful in other contexts, and which at this moment seems unable to cope with the problem of non-commutative field theories. These issues are currently under investigation and a more detailed account will appear elsewhere [27].

A   Appendix: Algebraic notes

We are considering non-commutative field theories with spacelike non-commutativity in four dimensions. The non-commutativity is given by

$$[x^i, x^j] = ib^{ij} \tag{34}$$

with b^{ij} antisymmetric and real.

It is well-known that functions are then multiplied using the Moyal product

$$f(x) \star g(x) = \exp(-ib^{ij}\partial_i\partial_j')f(x)g(x')|_{x'=x}. \tag{35}$$

One should be careful to consider what are allowable functions on the non-commutative space. The x are coordinates of the non-commutative space, and they generate the ring of functions. However, the x are not in the ring of allowable functions on the space, as they correspond to unbounded non-normalizable operators. We take the allowable functions to be the ring of formal power series in the x which are convergent and decay sufficiently fast at infinity, that is, a Schwarz space for the algebra generated by the x. For this to be a $\mathbb{C}^*$ algebra, we need to complete it. The functions can be represented by operators in a Hilbert space $\mathcal{H}$, and they are all compact. The completion is done with the following norm

$$\langle f, g \rangle = \int d^2x f^\star(x)g(x) = \text{tr}_{\mathcal{H}}(\hat{f}^\dagger \hat{g}) \tag{36}$$

where $\hat{f}, \hat{g}$ are the representative operators of the functions f, g. We require $|f| < \infty$ with the above norm, so the operators in the ring of functions are square integrable. In particular, the ring of operators contains all finite matrices.

We denote by G a finite Lie group with Lie algebra $\mathcal{G}$, which is finite dimensional. On a commutative space $\mathcal{M}$ we can consider tensoring the algebra of smooth functions on $\mathcal{M}$

with $\mathcal{G}$. We call these the infinitesimal gauge transformations. As the functions commute, the commutator of two infinitesimal gauge transformations closes on the Lie algebra of infinitesimal gauge transformations.

On a non-commutative space one can try to mimic this construction, but one runs into problems. Let us consider this possibility. The infinitesimal gauge transformations are given by sums of the form

$$\mathcal{L}(x) = \sum_a f_a(x)\mathcal{L}_a = f_a(x)\mathcal{L}_a \tag{37}$$

with $\mathcal{L}_a$ the generators of the Lie algebra $\mathcal{G}$. We think of $\mathcal{L}(x)$ as an operator acting on some Hilbert space and the Lie bracket is the commutator of two of these. We must interpret $\mathcal{L}$ in terms of the enveloping algebra of $\mathcal{G}$ in order to make sense of the product.

The commutator of two of these generators is given by

$$\begin{aligned}
[\mathcal{L}, \mathcal{L}'] &= f_a(x)\mathcal{L}_a f_b'(x)\mathcal{L}_b - f_b'(x)\mathcal{L}_b f_a(x)\mathcal{L}_a \tag{38}\\
&= \frac{1}{2}(f_a(x) \star f_b'(x) + f_b'(x) \star f_a(x))[\mathcal{L}_a, \mathcal{L}_b] \tag{39}\\
&\quad + \frac{1}{2}(f_a(x) \star f_b'(x) - f_b'(x) \star f_a(x))(\{\mathcal{L}_a, \mathcal{L}_b\}) \tag{40}
\end{aligned}$$

now, $[\mathcal{L}_a, \mathcal{L}_b] = f_{abc}\mathcal{L}_c$, so the first term in the above expression is in the Lie algebra, but the second one can only be in the Lie algebra if $\{\mathcal{L}_a, \mathcal{L}_b\} \in \mathcal{G}$. Usually this term belongs to the enveloping algebra of $\mathcal{G}$, but not to the Lie algebra and thus there is a technical problem in defining non-commutative gauge theories.

This problem may be resolved by using the enveloping algebra of $\mathcal{G}$ and requiring a finiteness condition on the representation. For simple $\mathcal{G}$ it acts on the field as the ring of $n \times n$ matrices for some n, and the gauge group is $U(n)$.[6]

Fields transform as irreducible representations of the Lie algebra either by matrix multiplication on the left or the right. Thus the only allowable representations have to be either left modules of the ring of $n \times n$ matrices tensored with the algebra of allowable functions, or they are right modules or bimodules of this algebra. That is, the fields transform in the fundamental, antifundamental or adjoint of the gauge group.

If we forego the $\mathcal{G}$ simple condition, then one can get product groups $\prod_i U(n_i)$. For each $U(n_i)$ factor a field can transform as a left module or a right module, but it cannot be a left module under two of the $U(n_i)$, as the x do not commute. Thus matter can only transform in adjoints under one group, and as left-right modules under two different gauge groups. This is exactly the type of spectrum that one can get for D-branes at singularities, so any non-commutative field theory is described by some quiver diagram (not necessarily associated to an orbifold). For each node in the quiver diagram we have an associated projector P_i which commutes with the Lie algebra of gauge transformations, and projects onto the direct summand associated to the node.

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[6]Other gauge groups may also be obtained, but they require a more complicated structure [28]

The gauge transformations thus act on fields as

$$\delta\phi = \mathcal{L}_L \star \phi - \phi \star \mathcal{L}_R \tag{41}$$

where L, R denotes the transformation under the gauge group operating on the left or right of the field ϕ belonging to the $(n_L, \bar{n}_R)$ representation of the gauge group, and the matrix multiplication is implicit.

The covariant derivative will be given by a commutator

$$D_j\phi = \partial_j\phi - [iA_j, \phi] \tag{42}$$

with A_j a Lie algebra valued allowable function. This is a compact way of writing

$$D_j\phi = \partial_j\phi - iA_{jL}\phi + i\phi A_{jR} \tag{43}$$

In terms of the projectors for the nodes in the quiver diagram associated to the field theory we have

$$P_L\phi = \phi P_R = \phi \tag{44}$$

for a field ϕ which corresponds to an arrow from the node associated to P_L to the node associated to P_R and also we have

$$A_L = P_L A = A P_L \tag{45}$$

for the connections and we can do similarly for gauge transformations.

Consider the case where the only non-trivial commutation relation is $[x^2, x^3] = ib$. The derivative operators in the $2, 3$-directions are equivalent to commutators

$$D_3\phi = \frac{1}{-ib}[\phi, x_2 + bA_3] \tag{46}$$

$$D_2\phi = \frac{1}{ib}[\phi, x_3 - bA_2] \tag{47}$$

The operators $\hat{x}^\mu = x^\mu + b^{\mu\nu}A_\nu$ are gauge covariant (they transform in the adjoint representation), which one may show explicitly by direct computation. The non-homogeneous term in the transformation of A is compensated because x does not commute with the gauge transformations. The commutator of these gives us

$$[\hat{x}_2, \hat{x}_3] = ib - ib^2 F_{23} \tag{48}$$

with F the curvature of the gauge group in the $2, 3$ directions, which is an allowable function. It can be shown that

$$[D_2, D_3](\phi) = i\left(F_{23}\phi - \phi F_{23}\right) \tag{49}$$

The variables $\hat{x}_j$ are the tools that we need to build gauge invariant observables.

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